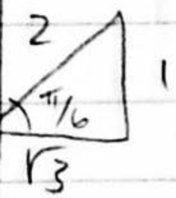
Ex: Compute the tangent line to the curve $\vec{R}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(2t) \rangle$ at $(\sqrt{3}, 1, 2)$.

Sol: Tangent function is:

$$\vec{R}'(t) = \langle -2\sin(t), 2\cos(t), -8\sin(2t) \rangle$$

To find the approximate time: solve

$$\vec{R}(t) = \langle \sqrt{3}, 1, 2 \rangle$$



ie $\begin{cases} 2\cos(t) = \sqrt{3} \\ 2\sin(t) = 1 \\ 4\cos(2t) = 2 \end{cases} \rightsquigarrow \begin{cases} \cos(t) = \frac{\sqrt{3}}{2} \\ \sin(t) = \frac{1}{2} \\ \cos(2t) = \frac{1}{2} \end{cases}$

try $t = \frac{\pi}{6}$:

$2\cos\left(\frac{\pi}{6}\right) = \sqrt{3}$ ✓

$2\sin(\pi/6) = 1$ ✓

$4\cos\left(\frac{2\pi}{6}\right) = 2$ ✓

$\rightsquigarrow$ ∴ the tangent vector at $(\sqrt{3}, 1, 2)$ is

$$\vec{R}'\left(\tfrac{\pi}{6}\right) = \langle -2\sin\left(\tfrac{\pi}{6}\right), 2\cos\left(\tfrac{\pi}{6}\right), -8\sin\left(\tfrac{2\pi}{6}\right) \rangle$$
$$= \langle -2\left(\tfrac{1}{2}\right), \tfrac{2\sqrt{3}}{2}, \tfrac{-8\sqrt{3}}{2} \rangle$$
$$= \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$$

∴ the desired tangent line has vector equation:

$$\vec{\ell}(t) = \vec{p} + t\,r'\left(\tfrac{\pi}{6}\right)$$

$$\boxed{\ell(t) = \langle \sqrt{3}, 1, 2 \rangle + t\langle -1, \sqrt{3}, -4\sqrt{3} \rangle}$$

§ 13.? : Arc Length

Last time: the arc length of a curve $\vec{R}(t)$ is given between $t=a$ and b is given by:

$$S = \int_{t=a}^{b} |\vec{R}'(t)|\,dt$$

from Calc II, the arc length was given by ($\vec{R}(t) = \langle x(t), y(t)\rangle$ on $a \le t \le b$):

$$S = \int_{t=a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$S = \int_{a}^{b} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$S = \int_{a}^{b} |R'(t)|\,dt$$

ex) compute the arc length of $\vec{R}(t) = \langle \cos(t), \sin(t), \ln(\cos(t))\rangle$ on $0 \le t \le \frac{\pi}{4}$.

$$S = \int_a^b |\vec{R}'(t)|\,dt \Rightarrow \int_0^{\pi/4} |R'(t)|\,dt$$

$$\vec{R}'(t) = \left\langle -\sin(t), \cos(t), \frac{-\sin(t)}{\cos(t)} \right\rangle = \langle -\sin(t), \cos(t), -\tan(t)\rangle$$

$$|\vec{R}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}$$

$$|\vec{R}'(t)| = \sqrt{1+\tan^2 t} = \sqrt{\sec^2(t)} = |\sec(t)|$$

positive on interval so, $= \sec(t)$

$|\vec{R}'(t)| = \sec(t)$ on $0 \le t \le \frac{\pi}{4}$

$\therefore s = \int_a^b |\vec{R}'(t)|\,dt = \int_0^{\pi/4} \sec(t)\,dt$

$s = \Big[\ln|\sec(t) + \tan(t)|\Big]_{t=0}^{\pi/4}$

$s = \Big[\ln|\sec(\frac{\pi}{4}) + \tan(\frac{\pi}{4})|\Big] - \Big[\ln|\sec(0) + \tan(0)|\Big]$

$s = \ln|\sqrt{2} + 1| - \ln\underbrace{|1 + 0|}_{0}$

$\boxed{s = \ln(1 + \sqrt{2})}$

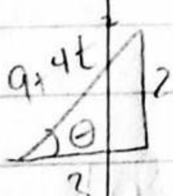
ex) Compute the arc length of $\vec{R}(t) = \langle 3\cos t, 3\sin t, t^2 \rangle$ on $2 \le t \le 10$

Sol $s = \int_a^b |\vec{R}'(t)|\,dt \Rightarrow s = \int_2^{10} |\vec{R}'(t)|\,dt$

$\vec{R}'(t) = \langle -3\sin(t), 3\cos(t), 2t \rangle$

$|\vec{R}'(t)| = \sqrt{(-3\cos(t))^2 + (3\cos t)^2 + (2t)^2} = \sqrt{9(\sin^2 t) + \cos^2 t + 4t^2}$

$= \sqrt{9 + 4t^2}$



$\frac{2t}{3} = \tan\theta$

$\frac{2}{3}\,dt = \sec^2(\theta)\,d\theta$

$\frac{\sqrt{9+4t^2}}{3} = \sec(t)$

$\therefore s = \int_2^{10} \sqrt{9 + 4t^2}\,dt = \int_{t=2}^{10} \frac{3\sqrt{9+4t^2}}{3} \cdot \frac{3}{2} \cdot \frac{2}{3}\,dt$

$s = \frac{9}{2} \int_2^{10} \sec\theta \sec^2\theta\,d\theta$

$$S = \frac{9}{2}\int_2^{10} \sec^3\theta \, d\theta$$

<u>to compute $\int \sec^3\theta \, d\theta$</u>

$$\int \sec^3\theta \, d\theta = \int \sec(\theta)\sec^2(\theta)\, d\theta = \int \sec\theta(1+\tan^2\theta)\, d\theta$$

$$= \int \sec\theta \, d\theta + \int \sec\theta \tan^2\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + \int \sec\theta \tan^2\theta \, d\theta$$

$u = \tan\theta \qquad dv = \sec\theta\tan\theta \, d\theta$

$du = \sec^2\theta \, d\theta \qquad v = \sec\theta$

$$\int u\,dv = uv - \int v\,du$$

$$= \sec\theta\tan\theta - \int \sec\theta\sec^2\theta \, d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta$$

$$\therefore \int \sec^3\theta \, d\theta = \ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta - \int \sec^3\theta \, d\theta$$

so, $$2\int \sec^3\theta = \ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta + C$$

$$\therefore \int \sec^3\theta \, d\theta = \frac{1}{2}\left[\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta\right] + C$$

hence, $$S = \frac{9}{2}\int_2^{10} \sec^3\theta \, d\theta$$

$$= \frac{9}{2}\left[\frac{1}{2}\left(\ln|\sec\theta + \tan\theta| - \sec\theta\tan\theta\right)\right]_2^{10}$$

$$s = \frac{9}{4}\left[\ln\left|\frac{\sqrt{9+4t^2}}{3} + \frac{2t}{3}\right| + \frac{\sqrt{9+4t^2}}{3}\cdot\frac{2t}{3}\right]_2^{10}$$

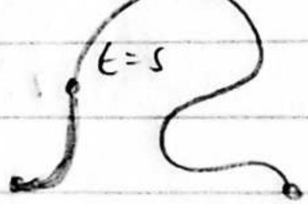
$$s = \frac{9}{4}\left[\left(\ln\left|\frac{\sqrt{409}}{3} + \frac{20}{3}\right| + \frac{20}{9}\sqrt{409}\right) - \left(\ln\left|\frac{5}{3}+\frac{4}{3}\right| - \frac{5\cdot 4}{9}\right)\right]$$

$$s = \frac{9}{4}\left[\ln\left|\frac{\sqrt{409}+20}{3}\right| + \frac{20\sqrt{409}}{9} - \ln(3) - \frac{20}{9}\right]$$

$$\boxed{s = 5\left(\sqrt{409}-1\right) + \frac{9}{4}\ln\left|\frac{20+\sqrt{409}}{9}\right|}$$

The arc length of a curve is a natural choice for parameter.

ie: we would like to parametrize $\vec{R}(t)$ so that at time $t=s$, the arc length (measured from some fixed point) is exactly s...



Define the arc length function for a parametrization by:

$$s(\beta) = \int_{\boxed{t=a}}^{\beta} |\vec{R}'(t)|\,dt$$

"some fixed point"

By FTC, $s'(\beta) = |\vec{R}'(\beta)|$

Moreover, s is an increasing function provided $|\vec{R}'(\beta)| \neq 0$ for all β, s is <u>strictly increasing</u>.

if a function is strictly increasing, it has an inverse function.

Next time: this guarantees a unit speed parametrization of $\vec{R}(t)$.